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Final Exam, Solutions to Exercise 2 and 4

Exercises 1 and 3 were solved satisfactorily by most participants, so that I am not including them here.

2. Uniqueness of semimartingale decomposition. (3 points) Consider two decompositions

$$X_t = X_0 + M_t + A_t = X_0 + \widetilde{M}_t + \widetilde{A}_t\,, \quad t \geq 0.$$

Then we get

$$M_t - \widetilde{M}_t = \widetilde{A}_t - A_t = \int_0^t \widetilde{a}_s - a_s ds\,, \quad t \geq 0,$$

so that $N := M - \widetilde{M}$ is both, a continuous local martingale (as difference of two continuous local martingales) and FV process. Proposition 3.11 from the lecture notes therefore implies that $N_t = N_0 = 0$ and hence $M = \widetilde{M}$. It follows that $A = \widetilde{A}$ and hence the uniqueness as claimed.

4. Volatility skew and Black Scholes model. (4 points) Key points of the answer:
a) Explanation/description:

- implied vol is computed from option prices via Black Scholes formula.
- implied volatility skew is a special pattern of implied vol as function of moneyness $M = (K/S_t)$, S_t the observed stock price: implied vol is strongly decreasing in M for $M < 1$, that is for $K < S_t$ (out of the money puts) has a min for $M \approx 1$ and is then *slightly* increasing for $M > K$.
- Skew is strongest for $T - t$ small.

b) Reasons for the skew and link to empirical deficiencies:

- Black Scholes is based on constant historical volatility, normally distributed log returns, continuous price paths (no jumps and crashes) and perfectly liquid markets, none of which fully holds in reality.
- In particular large price drops are much more likely in reality than in the model. This creates a huge demand in out of the money puts (crash insurance). At the same time traders are reluctant to sell these options since they are difficult to hedge (as jumps, illiquidity and stochastic vol may create a large tracking error).
- Combination of high demand and low supply gives a high price for put options with small K whoich translates to a high implied volatility.

Of course for full points I did not expect exactly this answer but something close to it.